We have so far considered R and its properties (including the Archimedean and density) and sequence limits. Our next concern is on nested intervals

By an interval A we mean a subset of R which can be expressed as one of the following types (with real numbers a, b and symbols 'positive infinity' and 'negative infinity' with a < b):

$$(a,b),\quad (a,b],\quad [a,b),\quad [a,b]$$

$$(a,+\infty),\quad [a,+\infty),\quad (-\infty,b),\quad (-\infty,b]$$

$$(-\infty,\infty) := \mathbb{R},$$

where we adopt the convention that $-\infty, \infty \notin \mathbb{R}$ & $-\infty < a < +\infty \quad \forall a \in A$.

Theorem (characterisation of intervals).

Let A be a nonempty subset of R. Then it is an interval if and only if it is (order-) convex in the sense that

(*) (x, y) is contained in A whenever x, y belong to A and x <y

(each element z of (a, b) can be represented as a convex combination of a and b, as
z = ta+(1-t)b with t: = (z-a)/(b-a) so the terminology)

Proof. Only need to show the sufficiency part; thus assume A satisfies (*). Separately consider four cases.

Case 1: A is bounded in the sense that A is bounded below and above. Hence a: =inf (A) and b: =sup(A) exist in R.
Then (a, b) is contained in A , and A is contained in [a,b], namely

$$(a,b) \subseteq A \subseteq [a,b]$$

Indeed, the 2nd inclusion is obvious by definitions of a and b. For the 1st let t be from (a,b): a<t<b. Then (WHY?) t is neither a lower bound nor an upper bound of A, and so there exist some x and y in A such that x<t<y; it follows from (*) that t belongs to A. Thus the displayed line is shown. Since [a,b] \ (a,b) consists exactly two points, I leave as an exercise for you to show then that A must be one of the intervals (a,b), (a,b], [a,b), [a,b].

Case 2: A is bounded below but not above. Then a: =inf A exists in R as before. Moreover

(a, positive infinity) is contained in A, and A is contained in [a, positive infinity), namely

$$(a,+\infty) \subseteq A \subseteq [a,+\infty) \qquad (2)$$

(so one of the inclusions is actually the equality because these sets differ at most by one point). Again we only need to prove the 1st inclusion in (2). For this, let

$$t \in (a,+\infty).$$

Since $a = \inf A < t$, $\exists\, x \in A$ s.t $x < t$.
Moreover, since A is not bounded above, t is not an upper bound of A

so $t < y$ for some $y \in A$. By the convexity assumption (*) and since $x < t < y$ with $x, y \in A$ it follows that $t \in A$. Thus (2) is shown.

Case 3: A is bounded above but not below. You are invited to supply your proof similar as for case 2.

Case 4. A is not bounded above nor below. Let z be any real number. Then
z is not an upper bound nor a lower bound of A and so x < z < y for some x, y in A and hence, by (*), z is in A (for real number z). Therefore. A = R.

The proof of this Theorem is complete

Nested interval theorem. Let

$$\{I_n : n \in \mathbb{N}\}$$

be a sequence of bounded closed nonempty intervals $I_n := [a_n, b_n]$ $(n \in \mathbb{N})$ with real numbers $a_n \le b_n$ such that

$$I_n \supseteq I_{n+1} \quad \forall\ n \in \mathbb{N} \qquad (\#)$$

(so, by MI, $I_n \supseteq I_k$ $\forall$ natural nos. $n < k$).

Then

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$$

i.e. $\exists\ x \in \mathbb{R}$ s.t. $x \in I_n\ \forall\ n \in \mathbb{N}$.

proof. By (#), $a_{n+1} \in I_{n+1} \subseteq I_n = [a_n, b_n]$

This implies that $a_n \le a_{n+1}$ and hence, by MI,

$$a_n \leq a_{n+j} \qquad \forall \text{ natural number } j$$

Similarly

$$b_{n+j} \leq b_n \quad \forall j \in \mathbb{N}$$

Since $a_{n+j} \leq b_{n+j}$ (as I_{n+j} nonempty), we have

$$a_n \leq a_{n+j} \leq b_{n+j} \leq b_n$$

and so $a_n \leq b_{n+j}$ & $a_{n+j} \leq b_n$

$$a_n \leq b_m \quad \forall m, n.$$

Consequently $a := \sup\{a_n : n \in \mathbb{N}\} \leq b_m \ \forall m$ &

$$a := \sup\{a_n : n \in \mathbb{N}\} \leq \underbrace{\inf\{b_m : m \in \mathbb{N}\}}_{\overset{\text{def}}{=} b}$$

Noting $a_n \leq a \leq b \leq b_n \ \forall n$,

$$\emptyset \neq [a, b] \subseteq \bigcap_{n \in \mathbb{N}} [a_n, b_n]. \quad \text{QED}.$$

Note. In fact

$$\emptyset \neq [a, b] = \bigcap_{n \in \mathbb{N}} [a_n, b_n], \qquad (\#)$$

because if $x \in \bigcap_{n \in \mathbb{N}} [a_n, b_n]$ then $a_n \leq x \leq b_n \ \forall n$

and so $\sup\{a_n : n \in \mathbb{N}\} \leq x \leq \inf\{b_n : n \in \mathbb{N}\}$.

Cor. Suppose additionally $\lim_n (b_n - a_n) = 0$.
(or, $\forall \varepsilon > 0, \exists n$ s.t. $b_n - a_n < \varepsilon$).
Then $a = b$, i.e. $\bigcap_{n \in \mathbb{N}} [a_n, b_n]$ is a singleton.

Pf. $0 \leq b - a \leq b_n - a_n \ \forall n$,

so $b - a = 0$ by Squeeze Th on limits, or by "Lemma for Making Life Easier".

2nd Proof of Nested Intervals Th (based on the Monotone Convergence Th). By a nest property of $I_n = [a_n, b_n]$ $(n \in \mathbb{N})$, we know that (a_n) is $\uparrow$ & bounded (by b_1) and $(b_n) \downarrow$ & bounded (below by a_1), so $a := \lim_n a_n$, $b := \lim_n b_n$ exist in $\mathbb{R}$ & $a_n \leq a \leq b \leq b_n$ $\forall n$ (Monotone Conv. Th. & order-preserving for limits), so

$$\emptyset \neq [a,b] \subseteq \bigcap_{n \in \mathbb{N}} [a_n, b_n]$$

(in fact the inclusion here can be replaced by $=$).